LSGNT Topic in Geometry: Exercises for Mapping Class Groups

For the following exercises, let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus.

1. Show that every homotopically essential simple closed curve on $\mathbb{T}^2$ is *homotopic* to a curve of the form $(a, b) \in \mathbb{R}^2/\mathbb{Z}^2$ such that $a, b \in \mathbb{Z}$ are coprime. Draw this curve for $(a, b) = (1, 3)$ and $(a, b) = (2, 3)$ on the standard embedding of $\mathbb{T}^2$ in $\mathbb{R}^3$. Show the same with homotopy replaced with *isotopy* (more difficult).

2. The purpose of this exercise is to prove that $\mathrm{MCG}(\mathbb{T}^2)$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$, and to find an explicit generating set for it. Here we consider $\mathrm{MCG}(\mathbb{T}^2)$ as the group of orientation-preserving homeomorphisms of $\mathbb{T}^2$ up to *homotopy.*

i) Show that two orientation-preserving homeomorphisms f and g of the torus are homotopic to each other if and only if they have the same induced map on $\pi_1(\mathbb{T}^2)$. (Hint: Use the fact that $\mathbb{T}^2$ is a $K(\pi, 1)$.)

ii) Use the isomorphism $\pi_1(\mathbb{T}^2) \cong H_1(\mathbb{T}^2; \mathbb{Z})$ to deduce that f and g are homotopic to each other if and only if they have the same induced map on $H_1(\mathbb{T}^2; \mathbb{Z})$.

iii) Note that the action of an orientation-preserving homeomorphism f of $\mathbb{T}^2$ preserves the intersection pairing on $H_1(\mathbb{T}; \mathbb{Z})$, and so represents as element of $\mathrm{Sp}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})$. Deduce that there is a well-defined homomorphism $\Phi \colon \mathrm{MCG}(\mathbb{T}^2) \to \mathrm{SL}(2, \mathbb{Z})$ that is injective.

iv) Show that the image of the Dehn twists about the standard curves $\alpha = (1, 0)$ and $\beta = (0, 1)$ are given by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Prove that the above matrices generate $\mathrm{SL}(2, \mathbb{Z})$, and conclude that Φ is surjective.

v) Conclude that Φ is an isomorphism, and that $\mathrm{MCG}(\mathbb{T}^2)$ is generated by the Dehn twists about α and β.

3. The purpose of this exercise is to prove the Nielsen–Thurston classification for $\mathrm{MCG}(\mathbb{T}^2) \cong \mathrm{SL}(2, \mathbb{Z})$. Let A be an element of $\mathrm{SL}(2, \mathbb{Z})$.

i) Show that the characteristic polynomial of A has the form $x^2 - \mathrm{tr}(A)\, x + 1 = 0$, where $\mathrm{tr}(A) \in \mathbb{Z}$ is the trace of the matrix A.

ii) Show that if $|\mathrm{tr}(A)| < 2$, then $A^{12} = id$.

iii) Show that if $|\mathrm{tr}(A)| = 2$, then A has 1 or -1 as eigenvalue. Use this to show that there is an essential simple closed curve on $\mathbb{T}^2$ that is invariant under A.

iv) Show that if $|\mathrm{tr}(A)| > 2$, then the eigenvectors of A have irrational slopes. Deduce that A does not fix the isotopy class of any essential simple closed curve on $\mathbb{T}^2$. In this case, A is called *Anosov.*

The proof of the Nielsen–Thurston classification for $g \geq 2$ is more complicated. In particular, there is no known embedding of the mapping class group in a linear group when $g \geq 3$.